A COUNTEREXAMPLE IN THE THEORY OF D-SPACES

DÁNIEL T. SOUKUP AND PAUL J. SZEPTYCKI

Abstract. Assuming $\diamondsuit$, we construct a T_2 example of a hereditarily Lindelöf space of size ω_1 which is not a D-space. The example has the property that all finite powers are also Lindelöf.

1. Introduction

The following notion is due to van Douwen, first studied with Pfeffer in [3].

Definition 1.1. *A T_1 space X is said to be a D-*space *if for each open neighbourhood assignment $\{U_x : x \in X\}$ there is a closed and discrete subset $D \subseteq X$ such that $\{U_x : x \in D\}$ covers the space.*

The question whether every regular Lindelöf space is D has been attributed to van Douwen [8]. Moreover, van Douwen and Pfeffer pointed out that

> "No satisfactory example of a space which is not a D-space is known, where by satisfactory example we mean an example having a covering property at least as strong as metacompactness or subparacompactness."

Indeed, the lack of satisfactory examples of D-spaces satisfying some interesting covering properties continues and there has been quite a bit of activity in the area in the last decades (see the surveys [5] and [8] for other related results and open problems). Whether regular Lindelöf spaces are D-spaces was listed as Problem 14 in Hrušák and Moore's list of 20 open problems in set-theoretic topology [10], and there are no consistency results in either direction even for hereditarily Lindelöf spaces. The question whether Lindelöf implies D for the class of T_1 spaces was also open and explicitly asked in [6] and more recently in [1].

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Corresponding author: Dániel T. Soukup.

In this note, assuming $\diamondsuit$, we construct an example of a hereditarily Lindelöf T_2 space that is not a D-space. The example also has the property that every finite power is Lindelöf, but we do not know if it can be made regular.

The article is structured as follows; in Section 2 we gather a few general facts and definitions, and in Section 3 we present the construction. In Section 4, we make some remarks and prove further properties of our construction. Finally, in Section 5 we state a few open problems.

2. Preliminaries

Delicate use of elementary submodels play crucial role in our arguments. We do not intend to give a precise introduction to this powerful tool since elementary submodels are widely used in topology nowadays; let us refer to [4]. However, we present here a few easy facts and a lemma which could serve as a warm-up exercise for the readers less involved in the use of elementary submodels.

Let $H(\vartheta)$ denote the sets which have transitive closure of size less than ϑ for some cardinal ϑ. The following facts will be used regularly without explicitly referring to them.

Fact 2.1. *Suppose that $M \prec H(\vartheta)$ for some cardinal ϑ and M is countable.*

(a) If $\mathcal{F} \in M$ and there is $F \in \mathcal{F} \setminus M$ then $\mathcal{F}$ is uncountable.
(b) If $B \in M$ and B is countable then $B \subseteq M$.

The next lemma is well-known, nonetheless we present a proof.

Lemma 2.2. *Let $\mathcal{F} \subseteq [\omega_1]^{<\omega}$ and suppose that M is a countable elementary submodel of $H(\vartheta)$ for some cardinal ϑ such that $\mathcal{F} \in M$. If there is an $F \in \mathcal{F}$ such that $F \notin M$ then there is an uncountable Δ-system $\mathcal{G} \subseteq \mathcal{F}$ in M with kernel $F \cap M$.*

Moreover, if $\psi(x, ...)$ is any formula with parameters from M and $\psi(F, ...)$ holds then $\mathcal{G}$ can be chosen in such a way that $\psi(G, ...)$ holds for every $G \in \mathcal{G}$, as well.

Proof. Suppose that $\mathcal{F}, M, F \in \mathcal{F} \setminus M$ and ψ is as above. Let $D = F \cap M$ and let $\mathcal{F}_0 = \{G \in \mathcal{F} : D \subseteq G \text{ and } \psi(G, ...)\}$. Clearly, $F \in \mathcal{F}_0 \in M$ and $F \notin M$ thus $\mathcal{F}_0$ is uncountable. Moreover, $F \cap \alpha = D$ for all α in a tail of $M \cap \omega_1$; that is, $\exists G \in \mathcal{F}_0 : G \cap \alpha = D$ and this holds in M as well, by elementary. Thus

$$M \models \exists \beta < \omega_1 \forall \alpha \in (\beta, \omega_1) \exists G \in \mathcal{F}_0 : G \cap \alpha = D.$$

Thus this holds in $H(\vartheta)$ as well, by elementary. Hence we can select inductively an uncountable Δ-system from $\mathcal{F}_0$. Using elementary again, there is such a Δ-system in M too.

$\square$

For any set-theoretic notion, including background on $\diamondsuit$, see [11].

There are different conventions in general topology whether to add regularity to the definition of a Lindelöf space. In this article, any topological space X is said to be *Lindelöf* iff every open cover has a countable subcover; that is, no separation is assumed.

Finally, we need a few other definitions.

Definition 2.3. *A collection $\mathcal{U}$ of subsets of a space X is called an* ω-cover *if for every finite $F \subseteq X$ there is $U \in \mathcal{U}$ such that $F \subseteq U$.*

Definition 2.4. *A collection $\mathcal{N}$ of subsets of a space X is called a* local π-network at the point x *if for each open neighbourhood U of x in X, there is an $N \in \mathcal{N}$ such that $N \subseteq U$ (it is not required that the sets in $\mathcal{N}$ be open, nor that they contain the point x).*

3. The Construction

We construct a topology by constructing a sequence $\{U_\gamma : \gamma < \omega_1\}$ of subsets of ω_1 such that $\gamma \in U_\gamma$ for every $\gamma \in \omega_1$. The example will be obtained by first taking the family $\{U_\gamma : \gamma < \omega_1\}$ as a subbasis for a topology on ω_1 and then refining it with a Hausdorff topology of countable weight.

The following lemma will be used to prove the Lindelöf property.

Lemma 3.1. *Consider a topology on ω_1 generated by a family $\{U_\gamma : \gamma < \omega_1\}$ as a subbase; sets of the form $U_F = \bigcap\{U_\gamma : \gamma \in F\}$ for $F \in [\omega_1]^{<\omega}$ form a base. If for every uncountable family $B \subseteq [\omega_1]^{<\omega}$ of pairwise disjoint sets there is a countable $B' \subseteq B$ such that*

$$|\omega_1 \setminus \bigcup\{U_F : F \in B'\}| \leq \omega$$

then the topology is hereditarily Lindelöf.

Proof. Fix an open family $\mathcal{U}$; we may assume that $\mathcal{U} = \{U_F : F \in \mathcal{G}\}$ for some $\mathcal{G} \subseteq [\omega_1]^{<\omega}$. Let M be a countably elementary submodel of $H(\vartheta)$ for some sufficiently large ϑ such that $\{U_\gamma : \gamma \in \omega_1\},\ \mathcal{G} \in M$.

It suffices to prove that $\bigcup\mathcal{U}$ is covered by the countable family $\mathcal{V} = \{U_F : F \in \mathcal{G} \cap M\}$. $\mathcal{V}$ clearly covers $\left(\bigcup\mathcal{U}\right) \cap M$ thus we consider an arbitrary $\alpha \in \left(\bigcup\mathcal{U}\right) \setminus M$. There is $G_0 \in \mathcal{G}$ such that $\alpha \in U_{G_0}$; let $F = G_0 \cap M$. If $F = G_0$ then $\mathcal{V}$ covers α thus we are done.

Otherwise there is an uncountable Δ-system $\mathcal{D} \subseteq \mathcal{G}$ in M with kernel F by Lemma 2.2. Consider the uncountable, pairwise disjoint family $B = \{G \setminus F : G \in \mathcal{D}\}$; by our hypothesis there is a countable $B' \subseteq B$ such that

$$|\omega_1 \setminus \bigcup\{U_H : H \in B'\}| \leq \omega.$$

B' can be chosen in M since $B \in M$; note that $B' \subseteq M$. Thus the countable set of points not covered also lie in M. Therefore, there is $G \setminus F \in B'$ such that $\alpha \in U_{G\setminus F}$. Hence $\alpha \in U_G$ since $\alpha \in U_{G_0} \subseteq U_F$ and $U_G = U_{G\setminus F} \cap U_F$. This completes the proof of the lemma. $\square$

Let us define now the topology which will be used to ensure the Hausdorff property.

Definition 3.2. *Define a topology on $[\mathbb{R}]^{<\omega}$ as follows. Let $Q \subseteq \mathbb{R}$ be a Euclidean open set and let $Q^* = \{H \in [\mathbb{R}]^{<\omega} : H \subseteq Q\}$. Sets of the form Q^* define a ρ topology on $[\mathbb{R}]^{<\omega}$.*

The proof of the following claim is straightforward.

Claim 3.3. *(1) $([\mathbb{R}]^{<\omega}, \rho)$ is of countable weight,*
(2) any family $\mathcal{X} \subseteq [\mathbb{R}]^{<\omega}$ of pairwise disjoint nonempty sets forms a Hausdorff subspace of $([\mathbb{R}]^{<\omega}, \rho)$.

Let us fix the countable base $\mathcal{W} = \{Q^* : Q$ is a disjoint union of finitely many intervals with rational endpoints$\}$ for $([\mathbb{R}]^{<\omega}, \rho)$.

For the remainder of this section we suppose $\diamondsuit$; thus $2^\omega = \omega_1$ and we can fix an enumeration

- $\{C_\alpha\}_{\alpha<\omega_1} = [\omega_1]^{\leq\omega}$ such that $C_\alpha \subseteq \alpha$ for all $\alpha < \omega_1$.

Also, there is a $\diamondsuit$-sequence $\{B_\gamma\}_{\gamma<\omega_1}$ on $[\omega_1]^{<\omega}$; that is,

- for every uncountable $B \subseteq [\omega_1]^{<\omega}$ there are stationary many $\beta \in \omega_1$ such that $B \cap [\beta]^{<\omega} = B_\beta$.

The next theorem is the key to our main result; we encourage the reader to first skip the quite technical proof of Theorem 3.4 and go to Corollary 3.8 to see how our main result is deduced. In particular, **IH**(3) assures the space is not a D-space and **IH**(4) makes the space hereditarily Lindelöf.

Theorem 3.4. *There exist $\{U_\gamma^\alpha\}_{\gamma\leq\alpha}$ and $\varphi_\alpha : (\alpha+1) \to [\mathbb{R}]^{<\omega}$ for $\alpha < \omega_1$ with the following properties:*

***IH**(1)* *$U_\gamma^\alpha \subseteq \alpha+1$ and $U_\alpha^\alpha = \alpha+1$ for every $\gamma \leq \alpha < \omega_1$, and the family $\varphi_\alpha[\alpha+1]$ is pairwise disjoint for every $\alpha < \omega_1$.*

***IH**(2)* *$U_\gamma^\alpha = U_\gamma^{\alpha_0} \cap (\alpha+1)$ and $\varphi_\alpha = \varphi_{\alpha_0} \restriction (\alpha+1)$ for all $\gamma \leq \alpha \leq \alpha_0$.*

Let τ_α denote the topology generated by the sets

$$\{U^\alpha_\gamma : \gamma \le \alpha\} \cup \{\varphi_\alpha^{-1}(W) : W \in \mathcal{W}\}$$

as a subbase. Let $U^\alpha_F = \bigcap\{U^\alpha_\gamma : \gamma \in F\}$ for $F \in [\alpha+1]^{<\omega}$.

***IH**(3) If C_α is τ_α closed discrete then $\bigcup\{U^\alpha_\gamma : \gamma \in C_\alpha\} \neq \alpha + 1$.*

***IH**(4) Let $T_\alpha = \{\beta \le \alpha : B_\beta$ is a pairwise disjoint family of finite subsets of β and there is a countable elementary submodel $M \prec H(\vartheta)$ for some sufficiently large ϑ such that (i)-(v) holds from below $\}$.*

(i) $M \cap \omega_1 = \beta$,

(ii) $\mathcal{W}, \{B_\gamma\}_{\gamma<\omega_1} \in M$,

(iii) there is a function $\varphi \in M$ such that $\varphi \restriction \beta = \varphi_\beta \restriction \beta$,

(iv) there is an uncountable $B \in M$ such that $M \cap B = B_\beta$, and

(v) there is a $\{V_\gamma\}_{\gamma<\omega_1} \in M$ such that $V_\gamma \cap \beta = U^\beta_\gamma \cap \beta$ for all $\gamma < \beta$.

Then

(a) if $\beta \in T_\alpha$ then B_β is a local π-network at β in τ_α,

(b) if $\beta \in T_\alpha \cap \alpha$ then for every $V \in \tau_\alpha$ with $\beta \in V$ the family

$$\{U^\alpha_F : F \in B_\beta, F \subseteq V\}$$

is an ω-cover of $(\beta, \alpha]$.

Proof. We prove by induction on $\alpha < \omega_1$ with inductional hypothesises **IH**(1)-**IH**(4)! Suppose we constructed $\{U^\beta_\gamma\}_{\gamma\le\beta}$ for $\beta < \alpha$. Let

$$U^{<\alpha}_\gamma = \cup\{U^\beta_\gamma : \gamma \le \beta < \alpha\} \text{ and } \varphi_{<\alpha} = \cup\{\varphi_\beta : \beta < \alpha\}.$$

Let τ^-_α denote the topology on α generated by the sets

$$\{U^{<\alpha}_\gamma : \gamma < \omega_1\} \cup \{\varphi^{-1}_{<\alpha}(W) : W \in \mathcal{W}\}$$

as a subbase. Let $U^{<\alpha}_F = \bigcap\{U^{<\alpha}_\gamma : \gamma \in F\}$ for $F \in [\alpha]^{<\omega}$. Note that if $\beta \in T_\alpha \cap \alpha$ then $\beta \in T_{\alpha'}$ for every $\alpha' \in (\beta, \alpha)$; hence **N**(i) and **N**(ii) below holds by **IH**(4):

N(i) for every $V \in \tau^-_\alpha$ with $\beta \in V$ the family

$$\{U^{<\alpha}_F : F \in B_\beta, F \subseteq V\}$$

is an ω-cover of (β, α);

N(ii) B_β is a local π-network at β in τ^-_α.

Therefore, it suffices to define $U^\alpha_\alpha = \alpha + 1$ and

$$U^\alpha_\gamma = \begin{cases} U^{<\alpha}_\gamma \text{ or} \\ U^{<\alpha}_\gamma \cup \{\alpha\} \end{cases}$$

and $\varphi_\alpha \restriction \alpha = \varphi_{<\alpha}$, $\varphi_\alpha(\alpha) = x_\alpha \in [\mathbb{R}]^{<\omega}$ such that

D(i) x_α is disjoint from $x_\beta = \varphi_{<\alpha}(\beta)$ for all $\beta < \alpha$,

D(ii) if $\beta \in T_\alpha \cap \alpha$ then for every $\beta \in V \in \tau_\alpha$ the family

$$\{U^\alpha_F : F \in B_\beta, F \subseteq V\}$$

is an ω-cover of $(\beta, \alpha]$,

D(iii) if C_α is τ^-_α closed discrete then $\alpha \notin U^\alpha_\gamma$ for all $\gamma \in C_\alpha$,

D(iv) if $\alpha \in T_\alpha$ then B_α is a local π-network at α in τ_α.

Case I. $T_\alpha \cap \alpha = \emptyset$

Let $U^\alpha_\alpha = \alpha + 1$, $U^\alpha_\gamma = U^{<\alpha}_\gamma$ for $\gamma < \alpha$. We proceed differently according to whether $\alpha \notin T_\alpha$ or $\alpha \in T_\alpha$.

Subcase A. $\alpha \notin T_\alpha$

Pick any $x_\alpha \in [\mathbb{R}]^{<\omega}$ disjoint from x_β for all $\beta < \alpha$. Clearly, **D**(i)-**D**(iv) are satisfied.

Subcase B. $\alpha \in T_\alpha$

It is clear that **D**(ii) and **D**(iii) are satisfied. Let M be a countable elementary submodel of $H(\vartheta)$ for some sufficiently large ϑ showing that $\alpha \in T_\alpha$. To find the appropriate $x_\alpha \in [\mathbb{R}]^{<\omega}$ we need that B_α is a local π-network at α in τ_α. Since $\{\varphi^{-1}_\alpha(W) : W \in \mathcal{W}, x_\alpha \in W\}$ will be a base at α in τ_α we need that for all $W \in \mathcal{W}$ such that $x_\alpha \in W$ there is an $F \in B_\alpha$ such that $\varphi[F] \subseteq W$. Since $\varphi[F] \subseteq W$ iff $\cup\varphi[F] \in W$, we need to find an accumulation point of the finite sets $\{\cup\varphi[F] : F \in B_\alpha\}$. We prove the following which will suffice:

Claim 3.5. *There is an $x_\alpha \in [\mathbb{R}]^{<\omega}$ such that $x_\alpha \cap x_\beta = \emptyset$ for all $\beta < \alpha$ and for all $W \in \mathcal{W}$ such that $x_\alpha \in W$ we have*

$$M \models |\{F \in B : \cup\varphi[F] \in W\}| > \omega.$$

Indeed, **D**(i) is satisfied. Let us check **D**(iv); clearly,

$$M \models \exists F \in B : \cup\varphi[F] \in W$$

for every $W \in \mathcal{W}$ such that $x_\alpha \in W$. Hence there is $F \in B \cap M = B_\alpha$ such that $\cup\varphi[F] = \cup\varphi_\alpha[F] \in W$, that is $\varphi_\alpha[F] \subseteq W$. Thus B_α is a local π-network at α in τ_α; that is, **D**(iv) is satisfied.

Proof of Claim 3.5. Since $M \models |B| > \omega$ there is $\widetilde{B} \in [B]^{\omega_1} \cap M$ and $k \in \omega, \{n_i : i < k\} \subseteq \omega$ such that $|F| = k$ for all $F \in \widetilde{B}$ and if $F = \{\gamma_i : i < k\}$ then $|\varphi(\gamma_i)| = n_i$ for all $i < k$.

Let $s = \sum_{i<k} n_i$. Now consider the pairwise disjoint s-element subsets $\{\cup\varphi[F] : F \in \widetilde{B}\}$ in $\mathbb{R}$. Clearly

$$M \models \big(\text{there are uncountably many pairwise disjoint } x \in [\mathbb{R}]^s \text{ such that } |\{F \in \widetilde{B} : \cup\varphi[F] \in W\}| > \omega \text{ for every } W \in \mathcal{W} \text{ with } x \in W\big).$$

Hence, there is $x_\alpha \in [\mathbb{R}]^s$ disjoint from x_β for all $\beta < \alpha$ such that $|\{F \in \widetilde{B} : \cup\varphi[F] \in W\}| > \omega$ for all $W \in \mathcal{W}$ with $x_\alpha \in W$. Thus

$$M \models |\{F \in \widetilde{B} : \cup\varphi[F] \in W\}| > \omega$$

which we wanted to prove. □

Case II. $T_\alpha \cap \alpha \neq \emptyset$

Let $T_\alpha \cap \alpha = \{\beta_n : n \in \omega\}$ and $\{G_n : n \in \omega\} \subseteq [\alpha]^{<\omega}$ such that for all $\beta \in T_\alpha \cap \alpha$ and $G \subseteq (\beta, \alpha)$ there are infinitely many $n \in \omega$ such that $\beta = \beta_n$ and $G = G_n$. Let $\{V_k(\beta) : k < \omega\}$ denote a decreasing neighbourhood base for the point $\beta \in T_\alpha \cap \alpha$ in τ_α^-. Note that $\{V_n(\beta_n) : n \in \omega, \beta_n = \beta\}$ is a base for $\beta \in T_\alpha \cap \alpha$.

Subcase A. $\alpha \notin T_\alpha$

We need the following claim:

Claim 3.6. *There is $F_n \in B_{\beta_n}$ for $n \in \omega$ such that*

A*(i)* $F_n \subseteq V_n(\beta_n)$,
A*(ii)* $G_n \subseteq U^{<\alpha}_{F_n}$,
A*(iii)* $F_n \cap C_\alpha = \emptyset$ *if C_α is τ_α^- closed discrete,*

Proof. There is $V \in \tau_\alpha^-$ such that $\beta_n \in V \subseteq V_n(\beta_n)$ and if C_α is closed discrete, then $C_\alpha \cap V \subseteq \{\beta_n\}$. The family

$$\{U_F^{<\alpha} : F \in B_{\beta_n}, F \subseteq V\}$$

is an ω-cover of (β_n, α) by $\mathbf{N}$(ii), thus there is $F_n \in B_{\beta_n}$ such that $F_n \subseteq V$ and $G_n \subseteq U^{<\alpha}_{F_n}$. □

Let $U^\alpha_\alpha = \alpha + 1$ and for $\gamma < \alpha$ let

$$U^\alpha_\gamma = \begin{cases} U^{<\alpha}_\gamma & \text{if } \gamma \notin \cup\{F_n : n \in \omega\}, \\ U^{<\alpha}_\gamma \cup \{\alpha\} & \text{if } \gamma \in \cup\{F_n : n \in \omega\}. \end{cases}$$

Pick any $x_\alpha \in [\mathbb{R}]^{<\omega}$ disjoint from x_β for all $\beta < \alpha$.

$\mathbf{D}$(i), $\mathbf{D}$(iii), and $\mathbf{D}$(iv) are trivially satisfied. Let us check $\mathbf{D}$(ii); fix $\beta \in T_\alpha \cap \alpha$, any neighbourhood $V \in \tau_\alpha$ such that $\beta \in V$, and a finite subset $G \subseteq (\beta, \alpha)$. We show that there is an $F \in B_\beta$, such that U^α_F covers $G \cup \{\alpha\}$ and $F \subseteq V$. There is $n \in \omega$ such that $\beta_n = \beta$, $G_n = G$, and $V_n(\beta_n) \subseteq V$; then $F_n \in B_{\beta_n}$ and $F = F_n$ does the job by $\mathbf{A}$(i), $\mathbf{A}$(ii) and the fact that $\alpha \in U^\alpha_{F_n}$.

Subcase B. $\alpha \in T_\alpha$

Let M be a countable elementary submodel of $H(\vartheta)$ for some sufficiently large ϑ showing that $\alpha \in T_\alpha$. Since $M \models |B| > \omega$ there is $\widetilde{B} \in [B]^{\omega_1} \cap M$ and $k \in \omega, \{n_i : i < k\} \subseteq \omega$ such that $|F| = k$ for all $F \in \widetilde{B}$ and if $F = \{\gamma_i : i < k\}$ then $|\varphi(\gamma_i)| = n_i$ for all $i < k$. Let $s = \sum_{i<k} n_i$. Enumerate α as $\{\alpha_n : n \in \omega\}$.

Claim 3.7. *There are $F_n \in B_{\beta_n}$ and $W_n \in \mathcal{W}$ for $n \in \omega$ such that*

B*(i)* $F_n \subseteq V_n(\beta_n)$,
B*(ii)* $G_n \subseteq U^{<\alpha}_{F_n}$,
B*(iii)* $F_n \cap C_\alpha = \emptyset$ *if* C_α *is* τ^-_α *closed discrete,*
B*(iv)* $W_n = \left(\cup\{Q_{n,i} : i < s\}\right)^*$ *is a basic open set of the topology* ρ *corresponding to* s *many disjoint rational intervals* $\{Q_{n,i} : i < s\}$ *of diameter less than* $\frac{1}{n}$,
B*(v)* $\overline{Q_{n+1,i}} \subseteq Q_{n,i}$ *for every* $i < s$ *in the Euclidean topology,*
B*(vi)* $\varphi(\alpha_n)$ *is disjoint from* $\cup\{Q_{n,i} : i < s\}$,
B*(vii)* *and finally*

$$M \models |\{F \in \widetilde{B} : F \subseteq \cap\{V_{F_k} : k \leq n\} \text{ and } \cup\varphi[F] \in W_n\}| > \omega.$$

Proof. We construct F_n and W_n by induction on $n \in \omega$. Suppose we constructed F_k and W_k for $k < n$ such that the hypothesises $\mathbf{B}$(i)-$\mathbf{B}$(vii) above are satisfied.

Let $D = \{F \in \widetilde{B} : F \subseteq \cap\{V_{F_k} : k < n\} \text{ and } \cup\varphi[F] \in W_{n-1}\}$ if $n > 0$ and $D = \widetilde{B}$ if $n = 0$; then $M \models |D| > \omega$. Just as in Claim 3.5

$$M \models \big(\text{there are uncountably many pairwise disjoint } x \in \mathbb{R}^s \cap W_{n-1} \text{ such that } |\{F \in D : \cup\varphi[F] \in W\}| > \omega \text{ for every } W \in \mathcal{W} \text{ with } x \in W\big).$$

Choose $x \in \mathbb{R}^s \cap W_{n-1}$ such that $|\{F \in D : \cup\varphi[F] \in W\}| > \omega$ for every $W \in \mathcal{W}$ with $x \in W$ and $x \cap \varphi(\alpha_n) = \emptyset$. Let $x = \{x_i : i < s\}$ and choose $W_n = \big(\cup\{Q_{n,i} : i < s\}\big)^*$ such that

- $Q_{n,i}$ is a rational interval of diameter less then $\frac{1}{n}$ for every $i < s$,
- $x_i \in Q_{n,i}$ for every $i < s$,
- $\overline{Q_{n,i}} \subseteq Q_{n-1,i}$ for every $i < s$ in the Euclidean topology (if $n > 0$),
- $\cup\{Q_{n,i} : i < s\} \cap \varphi(\alpha_n) = \emptyset$.

Let $D' = \big\{F \in \widetilde{B} : F \subseteq \cap\{V_{F_k} : k < n\}, \cup\varphi[F] \in W_n$ and $\beta_n < \min F\big\}$; clearly, $M \models |D'| > \omega$. Let $V \in \tau_\alpha^-$ such that $\beta_n \in V \subseteq V_n(\beta_n)$ and $V \cap C_\alpha \subseteq \{\beta_n\}$ if C_α is τ_α^- closed discrete. Applying $\mathbf{N}$(i) to $F \cup G_n$ for $F \in D' \cap M$ gives us that there is $F_n(F) \in B_{\beta_n}$ such that $F_n(F) \subseteq V$ and

$$U^{<\alpha}_{F_n(F)} \text{ covers } F \cup G_n$$

and hence $V_{F_n(F)}$ covers $F \cup G_n$. Thus

$$M \models \big(\text{ for every } F \in D' \text{ there is } F_n(F) \in B_{\beta_n} \text{ such that } F_n(F) \subseteq V \text{ and } V_{F_n(F)} \text{ covers } F \cup G_n.\big)$$

Finally, note that $M \models |B_{\beta_n}| \leq \omega$; thus

$$M \models \big(\text{there is } F_n \in B_{\beta_n} \text{ such that } F_n \subseteq V \text{ and } V_{F_n} \text{ covers } F \cup G_n \text{ for uncountably many } F \in D'\big).$$

It is now easily checked that F_n and W_n satisfies properties $\mathbf{B}$(i)-$\mathbf{B}$(vii). □

Let $U^\alpha_\alpha = \alpha + 1$ and for $\gamma < \alpha$ let

$$U^\alpha_\gamma = \begin{cases} U^{<\alpha}_\gamma & \text{if } \gamma \notin \cup\{F_n : n \in \omega\}, \\ U^{<\alpha}_\gamma \cup \{\alpha\} & \text{if } \gamma \in \cup\{F_n : n \in \omega\}. \end{cases}$$

Let $x_\alpha \in [\mathbb{R}]^{<\omega}$ be the unique s-element subset of $\mathbb{R}$ in the intersection $\cap\{\cup\{Q_{n,i} : i < s\} : n \in \omega\}$; existence and uniqueness follows from $\mathbf{B}$(iv) and $\mathbf{B}$(v), and x_α is disjoint from x_β for all $\beta < \alpha$ by $\mathbf{B}$(vi). Note that

$$\big\{\cap\{U^\alpha_{F_k} : k \leq n\} \cap \varphi_\alpha^{-1}(W_n) : n \in \omega\big\}$$

is a base for the point α in τ_α.

$\mathbf{D}$(i) is satisfied by $\mathbf{B}$(vi) and the fact that $x_\alpha \subseteq \cup\{Q_{n,i} : i < s\}$.

Let us check $\mathbf{D}$(ii); fix $\beta \in T_\alpha \cap \alpha$, any neighbourhood $V \in \tau_\alpha$ such that $\beta \in V$, and a finite subset $G \subseteq (\beta, \alpha)$. We show that there is an $F \in B_\beta$, such that U^α_F covers $G \cup \{\alpha\}$ and $F \subseteq V$. There is $n \in \omega$ such

that $\beta_n = \beta$, $G_n = G$, and $V_n(\beta_n) \subseteq V$; $F_n \in B_{\beta_n}$ does the job by **B**(i), **B**(ii) and the fact that $\alpha \in U^\alpha_{F_n}$.

D(iii) is satisfied by **B**(iii) and the definition of U^α_γ.

Finally, let us check **D**(iv); it suffices to show that for every $n \in \omega$ there is $F \in B_\alpha$ such

$$F \subseteq \cap\{U^\alpha_{F_k} : k \leq n\} \cap \varphi^{-1}_\alpha(W_n).$$

Condition **B**(vii) gives us this, using the observation that $\varphi_\alpha[F] \subseteq W$ iff $\cup\varphi_\alpha[F] \in W$ for any $F \in B_\alpha$ and $W \in \mathcal{W}$.

By all means, this completes the proof of the theorem.

□

Now we are ready to deduce our main result.

Corollary 3.8. *Suppose that $\{U^\alpha_\gamma\}_{\gamma \leq \alpha}$ and $\varphi_\alpha : (\alpha + 1) \to [\mathbb{R}]^{<\omega}$ for $\alpha < \omega_1$ are as in Theorem 3.4 and let $U_\gamma = \cup\{U^\alpha_\gamma : \gamma \leq \alpha < \omega_1\}$ for $\gamma < \omega_1$ and $\varphi = \cup\{\varphi_\alpha : \alpha < \omega_1\}$. Let τ denote the topology on ω_1 generated by the sets*

$$\{U_\gamma : \gamma < \omega_1\} \cup \{\varphi^{-1}(W) : W \in \mathcal{W}\}$$

as a subbase.

The space (ω_1, τ) is hereditarily Lindelöf, Hausdorff but not a D-space. Also, (ω_1, τ) has countable Ψ-weight.

Proof. First, we show that (ω_1, τ) is hereditarily Lindelöf and Hausdorff. We need the following observation.

Claim 3.9. *A Hausdorff topology of countable weight τ_{sc} refined by a hereditarily Lindelöf topology τ_{hl} on some set X is again a hereditarily Lindelöf, Hausdorff topology on X.*

Proof. Let τ_{ref} denote the topology generated by $\tau_{\text{sc}} \cup \tau_{\text{hl}}$ as a subbase; that is, τ_{ref} is the common refinement of τ_{sc} and τ_{hl}. τ_{ref} is clearly Hausdorff, we prove that for any open family $\mathcal{U} \subseteq \tau_{\text{ref}}$ there is a countable $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\cup\mathcal{U}_0 = \cup\mathcal{U}$. We can suppose that

$$\mathcal{U} = \{U_i \cap V^i_j : i \in \omega, j \in I_i\}$$

where $\{U_i : i \in \omega\} \subseteq \tau_{\text{sc}}$ and $\{V^i_j : i \in \omega, j \in I_i\} \subseteq \tau_{\text{hl}}$ for some index sets $\{I_i : i \in \omega\}$. For every $i \in \omega$ there is a countable $J_i \subseteq I_i$ such that

$$U_i \cap \bigcup\{V^i_j : j \in I_i\} = U_i \cap \bigcup\{V^i_j : j \in J_i\}$$

by the hereditarily Linelöfness of τ_{hl}. Thus

$$\cup\mathcal{U} = \bigcup\{U_i \cap V^i_j : i \in \omega, j \in J_i\}$$

which completes the proof. $\square$

Therefore, it suffices to prove that the topology generated by $\{U_\gamma : \gamma < \omega_1\}$ as a subbase on ω_1 is hereditarily Lindelöf. Lemma 3.1 and the proposition below gives us this result. Let $U_F = \bigcap\{U_\gamma : \gamma \in F\}$ for $F \in [\omega_1]^{<\omega}$.

Proposition 3.10. *For any uncountable family of pairwise disjoint sets $B \subseteq [\omega_1]^{<\omega}$, there is a countable $B' \subseteq B$ such that $\{U_F : F \in B'\}$ is a cover, moreover an ω-cover of a tail of ω_1.*

Proof. Fix some uncountable family $B \subseteq [\omega_1]^{<\omega}$ of pairwise disjoint sets. There is an $M \prec H(\vartheta)$ for some sufficiently large ϑ such that $B, \varphi, \{U_\gamma : \gamma < \omega_1\}, \{B_\gamma : \gamma < \omega_1\}, \mathcal{W} \in M$ and

$$M \cap \omega_1 = \beta \text{ and } B \cap M = B \cap [\beta]^{<\omega} = B_\beta.$$

We claim that $\bigcup\{U_F : F \in B'\}$ is an ω-cover of $\omega_1 \setminus (\beta + 1)$ for the countable $B' = B_\beta$. Indeed, fix some finite $K \subseteq \omega_1 \setminus (\beta + 1)$ and let $\alpha \in \omega_1 \setminus (\beta + 1)$ such that $K \subseteq \alpha$. Then $\beta \in T_\alpha$ ensured by the model M, and hence there is some $F \in B_\beta = B'$ such that $K \subseteq U_F^\alpha \subseteq U_F$ by **IH**(4). $\square$

Now we prove that (ω_1, τ) is not a D-space. Consider the neighbourhood assignment $\gamma \mapsto U_\gamma$; we show that $\cup\{U_\gamma : \gamma \in C\} \neq \omega_1$ for every closed discrete $C \subseteq \omega_1$. Since (ω_1, τ) is Lindelöf, $|C| \leq \omega$ and hence there is $\alpha < \omega_1$ such that $C_\alpha = C$. It suffices to note that C_α is τ_α closed discrete if τ closed discrete; indeed, then $\cup\{U_\gamma : \gamma \in C_\alpha\} \neq \alpha + 1$ by **IH**(3).

Finally, (ω_1, τ) has countable Ψ-weight since τ is a refinement of a Hausdorff topology which is of countable weight.

$\square$

4. Further properties

In [12] the authors asked the following:

Problem 4.1 ([12, Problem 4.6])**.** *Suppose that a space X has the property that for every open neighbourhood assignment $\{U_x : x \in X\}$ there is a second countable subspace Y of X such that $\bigcup\{U_x : x \in Y\} = X$ (*dually second countable*, in short). Is X a D-space?*

Our construction answers this question in the negative.

Proposition 4.2. *The space X constructed in Corollary 3.8 is dually second countable, however not a D-space.*

Proof. The space X has the property that every countable subspace is second countable; indeed, the subspace topology on $\alpha \in \omega_1$ is generated by the sets $U_\beta \cap \alpha$ for $\beta < \alpha$ and $\{\varphi^{-1}(W) : W \in \mathcal{W}\}$, using the notations of the previous section. Therefore, by the Lindelöf property, for every open neighbourhood assignment there is a countable and hence second countable subspace whose neighbourhoods cover the space. $\square$

Our aim now is to prove that the space constructed in Corollary 3.8 has the property that all its finite powers are Lindelöf. Indeed, by a theorem of Gerlits and Nagy [9], a space has all finite powers Lindelöf if and only if the space is an (ε)-space, i.e., every ω-cover has a countable ω-subcover.

Let us call our space from Corollary 3.8 X, and now establish the following theorem:

Theorem 4.3. *Every subspace of X is an (ε)-space.*

Proof. First, let us prove the following analogue of Lemma 3.1.

Lemma 4.4. *Consider a topology on ω_1 generated by a family $\{U_\gamma : \gamma < \omega_1\}$ as a subbase. If for every uncountable family $B \subseteq [\omega_1]^{<\omega}$ of pairwise disjoint sets there is a countable $B' \subseteq B$ such that*

$$\{U_F : F \in B'\} \text{ is an } \omega\text{-cover of a tail of } \omega_1$$

then the topology is a hereditarily (ε)-space.

Proof. Fix $Y \subseteq X$ and an ω-cover $\mathcal{U}$ of Y; we can suppose that $\mathcal{U} = \{\cup\{U_{F_i} : i < m\} : \{F_i : i < m\} \in \mathbb{F}\}$ for some $\mathbb{F} \subseteq \left[[\omega_1]^{<\omega}\right]^{<\omega}$. Let M be a countably elementary submodel of $H(\vartheta)$ for some sufficiently large ϑ such that $\{U_\gamma : \gamma \in \omega_1\}, \mathbb{F} \in M$. It suffices to prove the following.

Claim 4.5. *$M \cap \mathcal{U}$ is a countable ω-cover of Y.*

Proof. Let $K \in [Y]^{<\omega}$ and let $L = K \cap M$. Clearly, $M \cap \mathcal{U}$ covers K if $K = L \subseteq M$; thus, we can suppose that $K \neq L$ and hence $K \notin M$. There is some $\{F_i : i < m\} \in \mathbb{F}$ such that $K \subseteq \cup\{U_{F_i} : i < m\}$. Let $D_i = F_i \cap M$ for $i < m$ and we can suppose that there is some $n \leq m$ such that $F_i \neq D_i$ for $i < n$ and $F_i = D_i$ for $n \leq i < m$. It follows from Lemma 2.2 that there is an uncountable sequence $\{\{F_i^\alpha : i < m\} : \alpha < \omega_1\} \subseteq \mathbb{F}$ in M such that

(1) $\{F_i^\alpha : \alpha < \omega_1\}$ is an uncountable Δ-system with kernel D_i for every $i < n$,
(2) $F_i^\alpha = F_i$ for all $\alpha < \omega_1$ and $n \leq i < m$,
(3) $\beta \in U_{F_i}$ iff $\beta \in U_{F_i^\alpha}$ for every $\beta \in L$ and $\alpha < \omega_1$, $i < m$.

The uncountable family $\{F_i^\alpha \setminus D_i : \alpha < \omega_1\}$ is pairwise disjoint for every $i < n$. Hence if we let $F^\alpha = \bigcup_{i<n}(F_i^\alpha \setminus D_i)$ for $\alpha < \omega_1$ then there is $\Theta \in [\omega_1]^{\omega_1} \cap M$ such that $B = \{F^\alpha : \alpha \in \Theta\}$ is pairwise disjoint as well. By our hypothesis and elementary of M there is $J \in [\Theta]^\omega \cap M$ such that the countable $B' = \{U_{F^\alpha} : \alpha \in J\}$ is an ω-cover of a tail of ω_1; hence, an ω-cover of $\omega_1 \setminus M$ since finite sets which are not covered also lie in M. So there is $\alpha \in J$, and hence $\alpha \in M$, such that $K \setminus M \subseteq U_{F^\alpha} = \bigcap_{i<n} U_{F_i^\alpha \setminus D_i}$. The open set $U = \cup\{U_{F_i^\alpha} : i < m\}$ is in $\mathcal{U} \cap M$.

We claim that $K \subseteq U$. Fix $\beta \in K$; there is some $i < m$ such that $\beta \in U_{F_i}$. If $\beta \in L$ then $\beta \in U_{F_i^\alpha}$ by (3). Suppose that $\beta \in K \setminus M$; if $n \leq i < m$ then $F_i = F_i^\alpha$ and we are done. If $i < n$ then $\beta \in U_{F_i^\alpha \setminus D_i}$ and $\beta \in U_{F_i} \subseteq U_{D_i}$ so $\beta \in U_{F_i^\alpha}$. □

We are done with the proof of Lemma 4.4. □

We claim that Proposition 3.10 and Lemma 4.4 implies that X is a hereditarily (ε)-space. Indeed, our topology τ on ω_1 is generated by the sets

$$\{U'_\delta : \delta < \omega_1\} = \{U_\gamma : \gamma < \omega_1\} \cup \{\varphi^{-1}(W) : W \in \mathcal{W}\}$$

as a subbase. Suppose that $B \subseteq [\omega_1]^{<\omega}$ is an uncountable family of pairwise disjoint sets; since $\mathcal{W}$ is countable, there is $B_0 \in [B]^{\omega_1}$ such that $U'_\delta \in \{U_\gamma : \gamma \in \omega_1\}$ for $\delta \in \cup B_0$. Thus by Proposition 3.10, there is some countable $B' \subseteq B_0$ such that $\{U'_F : F \in B'\}$ is an ω-cover of a tail of ω_1. Hence the assumption of Lemma 4.4 holds for X, thus X is a hereditarily (ε)-space. □

A well known weakening of $\diamondsuit$ is Ostaszewski's $\clubsuit$, which is known to be consistent with $\omega_1 < 2^\omega$. We remark that $\clubsuit$ is not enough to construct a space of size ω_1 which is Hausdorff and Lindelöf but not a D-space.

Claim 4.6. *It is consistent that $\clubsuit$ holds, 2^ω is arbitrarily large, and every T_1 Lindelöf space of size less than 2^ω is a D-space.*

Proof. It is known that T_1 Lindelöf spaces of size less than the dominating number $\mathfrak{d}$ are Menger, and L. Aurichi proved that every Menger space is a D-space [2]. Thus, it suffices to show that there is a model of ZFC where $\clubsuit$ holds, 2^ω is arbitrarily large, and $\mathfrak{d} = 2^\omega$. I. Juhász proved in an unpublished note that it is consistent that $\clubsuit$ holds, 2^ω is arbitrarily large, and Martin's Axiom holds for countable posets; for a proof see [7]. It is easy to see that Martin's Axiom for countable posets imply $\mathfrak{d} = 2^\omega$. □

5. Questions

Let us state some questions concerning Theorem 4.3. We do not know whether the analogue of Claim 3.9 holds for hereditarily (ε)-spaces.

Question 5.1. *Suppose that τ and σ are second countable and hereditarily (ε)-space topologies respectively on some set X. Is the topology generated by $\tau \cup \sigma$ a hereditarily (ε)-space again?*

We do not know if being a hereditarily (ε)-space implies the hereditarily Lindelöfness of finite powers.

Question 5.2. *Suppose that a space X is a hereditarily (ε)-space. Is X^n hereditarily Lindelöf for all $n \in \omega$?*

The following might be easier, nonetheless seems to be open.

Question 5.3. *Suppose that a space X has the property that A^2 is Lindelöf for all $A \subseteq X$. Is X^2 hereditarily Lindelöf?*

We mention two other versions of the question above.

Question 5.4.
(i) Suppose that a space X has the property that $A \times B$ is Lindelöf for all $A, B \subseteq X$. Is X^2 hereditarily Lindelöf?
(ii) Suppose that the spaces X, Y have the property that that $A \times B$ is Lindelöf for all $A \subseteq X$ and $B \subseteq Y$. Is $X \times Y$ hereditarily Lindelöf?

Of course, the main interest is in obtaining a regular counterexample to van Douwen's question. We conjecture that one should be able to modify our construction in such a way that the sets $\{U_\gamma : \gamma \in \omega_1\} \cup \{\omega_1 \setminus U_\gamma : \gamma \in \omega_1\}$ generate a 0-dimensional, T_1 topology that is not a D-space and has some additional interesting covering properties. E.g.,

Question 5.5. *Can we modify the construction to obtain a 0-dimensional T_1 (hence regular) Lindelöf non D-space?*

Finally, let us finish with a more general question. We believe that our construction can be modified so that its finite powers are hereditarily Lindelöf, thus its ωth power as well.

Question 5.6. *Suppose that a regular space X has the property that X^ω is hereditarily Lindelöf. Is X a D-space?*

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Institute of Mathematics, Eötvös Loránd University, Budapest, Hungary

E-mail address: daniel.t.soukup@gmail.com

Department of Mathematics and Statistics, York University, Toronto, ON M3J 1P3 Canada

E-mail address: szeptyck@yorku.ca